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Problem of Hidden Variables

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The problem of hidden variables in quantum mechanics is formalized as follows. A general or contextual (noncontextual) hidden-variables theory is defined as a mapping $f: Q \times M \to C$ ($f: Q \to C$) where Q is the set of projection operators in the appropriate (quantum) Hilbert space, M is the set of maximal Boolean subalgebras of Q and C is a (classical) Boolean algebra. It is shown that contextual (noncontextual) hidden-variables always exist (do not exist).

The purpose of this note is to state clearly the problem of hidden variables in quantum mechanics. This problem was formally posed by von Neumann (1955) sixty years ago, but a lot of confusion existed until the clarifying paper by Bell (1966). In my opinion, however, the situation is not yet clear and Bell's (1965) theorem is frequently misinterpreted. Consequently, I think that it is worth trying a new approach in order to define sharply the nature of the hidden-variable problem, using the language of quantum logics. The main new aspects of the approach are (1) its generality, (2) the connection between classical and quantum logics through logics associated with experiments rather than directly, and (3) the definition of contexts as algebras of quantum propositions.

Quantum mechanics states that the basic mathematical structure for the study of a physical system is a Hilbert space H. Self-adjoint operators on H represent observables and self-adjoint, trace class, positive operators (density operators) represent states. In the quantum logic approach it is assumed that all observables can be obtained from dichotomic ones, which are associated to idempotent operators (projectors). Following Birkhoff and von Neumann (1936), I shall identify projectors with propositions, these being the basic elements in the quantum logic approach. Then, by a standard method (Birkhoff and von Neumann, 1936; Jauch, 1968), the set Q of all propositions making sense for a given physical system is endowed with the structure

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of a non-Boolean orthomodular lattice and states are probability measures on Q. Let S be the set of states so defined (which can be put in one-to-one correspondence with the set of density operators on H). The mathematical structure of Q and S fully defines the quantum formalism as applied to the physical system (except for the laws of evolution) without further need of using explicitly the Hilbert space.

The hidden-variables problem can be posed as the problem of finding another set C of (hidden-variable) propositions and a corresponding set of mappings of C into $[0, 1]$, which define the (hidden-variable) states. I think that everybody will agree with this statement of the problem. What seems controversial is the relation between Q and C and between the associated sets of states. For the moment I shall make only the minimal assumption that the set C is an orthocomplemented poset admitting probability measures, but I shall add that any hidden-variable theory should fulfil the following two conditions:

A. Every hidden-variable state can be obtained as a convex combination of dispersion-free states, these being mappings of C into $\{0, 1\}$.
B. The predictions of the hidden-variable theory must agree with those of quantum mechanics for all possible experiments.

Instead of condition A, or in addition to it, one might demand that C is (or can be embedded into) a Boolean algebra, as a condition for the hidden-variable theory being classical, but I shall not use this assumption here.

In order to deal properly with condition B, I shall begin by formalizing the concept of an *experiment*. As usual, I shall consider that it consists of the *preparation* of a physical system followed by a *measurement*. (Between the preparation and the measurement there is a time interval during which the system evolves, but this is irrelevant here and I shall ignore the *evolution*.) I also assume that the preparation defines a *state*. On the other hand, in order to formalize the concept of measurement, we must take into account that two experiments in which we prepare the same state and measure the same set of propositions should not be considered as essentially different, even if the experimental setup is quite diverse. (By measuring a proposition I mean finding the probability that the proposition is true in the given—in general mixed—state.) In fact, both experiments should provide the same results according to the scientific principle of the reproducibility of experiments. Therefore, if M is the set of possible measurements, $m_1 \in M$ and $m_2 \in M$ are essentially identical whenever the same set of propositions is measured in m_1 as in m_2. This leads us to identify tentatively measurements with subsets of Q. There are two constraints, however. In the first place, not all propositions can be simultaneously measured. Those which can are called *compatible*, a concept that I shall define (Jauch, 1968) by the condition that these propositions belong to the same Boolean subalgebra of Q. (The corresponding projectors on the Hilbert space commute pairwise.) In the second place, if the subset of Q associated to the measurement m_1 contains the one associated to m_2, the two measurements should not be considered as different, because performing m_2 is equivalent to performing m_1 without looking at the results obtained for the propositions contained in m_1 and not in m_2. All this leads to the following:

Definition. A *measurement* $m \in M$ is a maximal Boolean subalgebra of Q, i.e., there is no bigger Boolean subalgebra of Q containing m.

Therefore, M is the set of maximal Boolean subalgebras of Q.

Now we can define the most general hidden-variable theory that includes the requirements A and B stated above:

Definition. A *hidden-variable theory* is a mapping $f: Q \times M \to C$, the domain of which is the set of pairs $\{(a, m)\}$ such that $a \in m$, fulfilling the following conditions:

A1. The condition

$$s(a) = \sum \lambda_k p_k(f(a, m)), \qquad \forall m \in M, \quad \forall a \in m, \quad \forall s \in S \tag{1}$$

holds, where $\{p_k\}$ is the set of dispersion-free states on C, and the real numbers λ_k fulfil

$$\lambda_k \geq 0, \qquad \sum \lambda_k = 1 \tag{2}$$

B1. The image of every Boolean subalgebra $m \in M$ is a Boolean subalgebra, and the one-variable mapping $f_m: m \to C$, where $f_m(a) = f(a, m)$, is an isomorphism for any fixed m.

The reason for Axiom B1 is that the set of propositions measurable in a particular experiment can be endowed naturally with the structure of a Boolean algebra (Pykacz and Santos, 1990), which we should assume isomorphic to the associated subalgebra of quantum propositions.

This definition of hidden-variable theories corresponds to quite general ones, usually called *contextual* because the probability for a proposition $a \in Q$ being true depends on the context of the measurement, that is, on $m \in M$ in our notation, rather than being a property of the proposition and the state $s \in S$ alone. Bell (1966) showed, with an informal argument, that contextual theories of quantum mechanics cannot be excluded, and Gudder (1970) gave a more formal proof. In our approach we can prove the following.

Theorem. A contextual hidden-variable theory exists for every state of any physical system.

I shall give the proof by constructing explicitly a contextual hidden-variable theory for any state of an arbitrary physical system. Given the quantum lattice Q, we define C to be a horizontal sum (Pykacz and Santos, 1990; Kalmbach, 1983) of disjoint Boolean algebras which are isomorphic to the elements of M (i.e., to a maximal Boolean subalgebra of Q). It can be easily proved (Kalmbach, 1983) that a horizontal sum of Boolean algebras is an orthomodular lattice. The isomorphisms define the mapping f. [Note that a given proposition $a \in Q$ may possess several different images $f(a, m) \in C$, one for each maximal Boolean subalgebra of Q containing a.] Any quantum state $s \in S$ induces a state p on C by defining

$$p(f(a, m)) = s(a) \tag{3}$$

It can be seen that both conditions A and B stated at the beginning for hidden-variable theories are fulfilled in this construction.

Contextual hidden-variable theories are unsatisfactory in many respects. In particular, they are nonlocal, in Bell's (1965) sense, as can be shown without difficulty. In consequence, it is interesting to define another, more restricted, family as follows:

Definition. A *noncontextual* hidden-variable theory is a mapping $f: Q \to C$ such that properties analogous to (1) and (2) hold, i.e.:

A2. The following condition holds:

$$s(a) = \sum \lambda_k p_k(f(a)), \qquad \forall a \in Q, \quad \forall s \in S, \quad \lambda_k \geq 0, \quad \sum \lambda_k = 1 \tag{4}$$

B2. The image of every Boolean subalgebra $m \in M$ is a Boolean subalgebra, and the restriction of the mapping to m is an isomorphism.

Noncontextual hidden-variable theories, as defined above, are not possible in general. The first proof of impossibility was given by Gleason (1957). Bell's (1965) theorem is also a valid proof, as the condition of locality is weaker than noncontextuality.

A fundamental property of noncontextual hidden-variable theories is the following:

Theorem. If $a, b, c, \ldots, d \in Q$ are such that a is compatible with b, b with $c, \ldots$, and d with a (not necessarily for other pairs), then for any state $s \in S$ the following inequality holds:

$$\sigma(a, b) + \sigma(b, c) + \cdots \geq \sigma(d, a) \tag{5}$$

where

$$\sigma(a, b) = s(a) + s(b) - 2s(a \wedge b) \tag{6}$$

$a \wedge b$ is the meet of the propositions a and b.

I omit the proof, which is similar to the proof of the main theorem in Pykacz and Santos (1991). From this theorem, the impossibility of noncontextual hidden-variable theories follows easily by exhibiting a particular example of a quantum mechanical state violating (5). This parallels the standard proof of Bell's theorem (Bell, 1966).

Actually, Bell studied another family of hidden-variable theories which he called local. Local theories are extremely interesting from a conceptual point of view, but it is difficult to characterize them within the quantum logic approach, because locality is related to space-time and not to the mathematical structure of the set of propositions. For this reason I do not study them here. I only point out that they are partially contextual.

I want to stress that all impossibility proofs of noncontextual or local hidden-variable theories make implicit use of the assumption that *all* self-adjoint operators in the Hilbert space of a physical system correspond to empirically observable quantities (except for superselection rules) or, equivalently, that *all* density operators correspond to states realizable in the laboratory. If we do not make use of such an assumption, it is still an open question whether noncontextual hidden-variable theories of quantum mechanics in perfect agreement with experiments are possible. [The received wisdom is that local—and, *a fortiori*, noncontextual—hidden-variable theories have been already refuted empirically, but this belief is wrong (Ferrero *et al.*, 1990; Santos, 1991).]

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